

# Logarithmic Sobolev trace inequalities

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## Abstract

We prove a logarithmic Sobolev trace inequality in a gaussian space and we study the trace operator in the weighted Sobolev space  $W^{1,p}(\Omega, \gamma)$  for sufficiently regular domain. We exhibit examples to show the sharpness of the results. Applications to PDE are also considered.

## 1 Introduction

Sobolev Logarithmic inequality states that

$$\int_{\mathbb{R}^N} |u|^p \log |u| d\gamma \leq \frac{p}{2} \int_{\mathbb{R}^N} |\nabla u|^2 |u|^{p-2} \text{sign } u d\gamma + \|u\|_{L^p(\mathbb{R}^N, \gamma)}^p \log \|u\|_{L^p(\mathbb{R}^N, \gamma)}, \quad (1.1)$$

where  $1 < p < +\infty$ ,  $\gamma$  is the Gauss measure and  $L^p(\mathbb{R}^N, \gamma)$  is the weighted Lebesgue space (see §2 for the definitions). This inequality was first proved in [17] (see also [3] for more general probability measure). It has many applications in quantum field theory and differently from classical Sobolev inequality it is independent of dimension and easily extends to the infinite dimension.

In terms of functional spaces inequality (1.1) implies the imbedding of weighted Sobolev space  $W^{1,p}(\mathbb{R}^N, \gamma)$  into the weighted Zygmund space  $L^p(\log L)^{\frac{1}{2}}(\mathbb{R}^N, \gamma)$ . The imbedding holds also for  $p = 1$  and it is connected with gaussian isoperimetric inequality and symmetrization (see [19], [14] and [23]).

For  $p = +\infty$  one obtains (see [21] and [1]) that if  $u$  is a Lipschitz continuous function, then  $u \in L^\infty(\log L)^{-\frac{1}{2}}(\mathbb{R}^N, \gamma)$ .

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This kind of imbeddings are also studied in [8] in the more general case of rearrangement-invariant spaces.

In [10] a set  $\Omega \subseteq \mathbb{R}^N$  and the space  $W_0^{1,p}(\Omega, \gamma)$  are considered using properties of rearrangements of functions; the authors prove that if  $u \in W_0^{1,p}(\Omega, \gamma)$  with  $1 \leq p < +\infty$ , then  $u \in L^p(\log L)^{\frac{1}{2}}(\Omega, \gamma)$  and

$$\|u\|_{L^p(\log L)^{\frac{1}{2}}(\Omega, \gamma)} \leq C_1 \|\nabla u\|_{L^p(\Omega, \gamma)}. \quad (1.2)$$

Moreover if  $u$  is lipschitz continuous function with  $\lim_{x \in \Omega, |x| \rightarrow +\infty} u(x) = 0$  and  $u|_{\partial\Omega} = 0$ , then  $u \in L^\infty(\log L)^{-\frac{1}{2}}(\Omega, \gamma)$  and

$$\|u\|_{L^\infty(\log L)^{-\frac{1}{2}}(\Omega, \gamma)} \leq C_2 \|\nabla u\|_{L^\infty(\Omega)}. \quad (1.3)$$

The constants  $C_1, C_2$  depend only on  $p$  and  $\gamma(\Omega)$ . Analogue inequalities have been obtained in infinite dimensional case and in the Lorentz-Zygmund spaces (see the appendix of [16]).

A first result of our paper is to obtain (1.2) when  $u \in W^{1,p}(\gamma, \Omega)$  (see §3); in this case, as one can expect, smoothness assumption on  $\partial\Omega$  have to be made. Besides the continuity also the compactness of the imbedding of  $W^{1,p}(\Omega, \gamma)$  in a Zygmund space is studied. As a consequence we obtain a Poincaré-Wirtinger type inequality. We analyze also the case  $p = +\infty$ . These results are sharp and counterexamples in this direction are given. Applications of these results to PDE are also considered.

The results explained above are used to investigate Sobolev trace inequalities. This kind of inequalities play a fundamental role in problems with nonlinear boundary conditions. In the euclidean case the Sobolev trace inequality (cf. e.g. [18]) tell us that if  $\Omega$  is smooth enough and  $1 \leq p < N$ , then there exists a constant  $C$  (depending only on  $\Omega$  and on  $p$ ) such that

$$\|Tu\|_{L^{\frac{p(N-1)}{N-p}}(\partial\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)} \quad \text{for every } u \in W^{1,p}(\Omega),$$

where  $T$  is the trace operator. This kind of inequalities has been developed via different methods and in different settings by various authors including Besov [6], Gagliardo [11], Lions and Magenes [22]. Trace inequality that involves rearrangement-invariant norms are considered in [?].

To investigate about trace operator in the weighted Sobolev space  $W^{1,p}(\Omega, \gamma)$  in §4 we need a Sobolev trace inequality. We prove that if  $\Omega$  is a smooth domain and  $u \in C^\infty(\overline{\Omega})$  then

$$\int_{\partial\Omega} |u|^p \log^{\frac{p}{2p'}}(2 + |u|) \varphi dS \leq C \|u\|_{W^{1,p}(\Omega, \gamma)}^p. \quad (1.4)$$

This inequality is sharp and captures the spirit of the Gross inequalities: the logarithmic function replaces the powers in this case too. We analyze also the case  $p = +\infty$ .

Using (1.4), we can define the trace operator and to prove continuity and compactness of the operator  $W^{1,p}(\Omega, \gamma)$  into  $L^p(\partial\Omega, \gamma)$  for sufficiently regular domain  $\Omega \subseteq \mathbb{R}^N$ . Moreover we prove a Poincaré trace inequality is obtained in a suitable subspace of  $W^{1,p}(\Omega, \gamma)$ . We give also some applications of these results to PDE.

Another Sobolev trace inequality is obtained in [24] as limit case of the classical trace Sobolev inequality.

## 2 Preliminaries

In this section we recall some definitions and results which will be useful in the following.

### 2.1 Gauss measure and rearrangements

Let  $\gamma$  be the  $N$ -dimensional Gauss measure on  $\mathbb{R}^N$  defined by

$$d\gamma = \varphi(x) dx = (2\pi)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{2}\right) dx, \quad x \in \mathbb{R}^N$$

normalized by  $\gamma(\mathbb{R}^N) = 1$ .

We will denote by  $\Phi(\tau)$  the Gauss measure of the half-space  $\{x \in \mathbb{R}^N : x_N < \tau\}$  :

$$\Phi(\tau) = \gamma(\{x \in \mathbb{R}^N : x_N < \tau\}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\tau} \exp\left(-\frac{t^2}{2}\right) dt \quad \forall \tau \in \mathbb{R} \cup \{-\infty, +\infty\}.$$

We define the decreasing rearrangement with respect to Gauss measure (see e.g. [13]) of a measurable function  $u$  in  $\Omega$  as the function

$$u^{\otimes}(s) = \inf \{t \geq 0 : \gamma_u(t) \leq s\} \quad s \in ]0, 1],$$

where  $\gamma_u(t) = \gamma(\{x \in \Omega : |u| > t\})$  is the distribution function of  $u$ .

### 2.2 Sobolev and Zygmund space

The weighted Lebesgue space  $L^p(\Omega, \gamma)$  is the space of the measurable functions  $u$  on  $\Omega$  such that  $\int_{\Omega} |u|^p d\gamma < +\infty$ . We recall also that the weighted Sobolev space  $W^{1,p}(\Omega, \gamma)$  for  $1 \leq p < +\infty$  is defined as the space of the

measurable functions  $u \in L^p(\Omega, \gamma)$  such that there exists  $g_1, \dots, g_N \in L^p(\Omega, \gamma)$  that verify

$$\int_{\Omega} u \frac{\partial}{\partial x_i} \psi \varphi - \int_{\Omega} u \psi x_i \varphi = \int_{\Omega} g_i \psi \varphi \quad i = 1, \dots, N \quad \forall \psi \in D(\Omega).$$

We stress that  $u \in W^{1,p}(\Omega, \gamma)$  is a Banach space with respect to the norm  $\|u\|_{W^{1,p}(\Omega, \gamma)} = \|u\|_{L^p(\Omega, \gamma)} + \|\nabla u\|_{L^p(\Omega, \gamma)}$ .

The Zygmund space  $L^p(\log L)^\alpha(\Omega, \gamma)$  for  $1 \leq p \leq +\infty$  and  $\alpha \in \mathbb{R}$  is the space of the measurable functions on  $\Omega$  such that the quantity

$$\|u\|_{L^p(\log L)^\alpha(\Omega, \gamma)} = \begin{cases} \left( \int_0^{\gamma(\Omega)} [(1 - \log t)^\alpha u^\otimes(t)]^p \right)^{\frac{1}{p}} & \text{if } 1 \leq p < +\infty \\ \sup_{t \in (0, \gamma(\Omega))} [(1 - \log t)^\alpha u^\otimes(t)] & \text{if } p = +\infty \end{cases} \quad (2.1)$$

is finite. The space  $L^p(\log L)^\alpha(\Omega, \gamma)$  is not trivial if and only if  $p < +\infty$  or  $p = +\infty$  and  $\alpha \leq 0$ .

The Zygmund spaces are the natural spaces in the context of Gauss measure, because of the following property of isoperimetric function is (see [20]):

$$\varphi_1 \circ \Phi^{-1}(t) \sim t(2 \log \frac{1}{t})^{\frac{1}{2}} \quad \text{for } t \rightarrow 0^+ \text{ and } t \rightarrow 1^-. \quad (2.2)$$

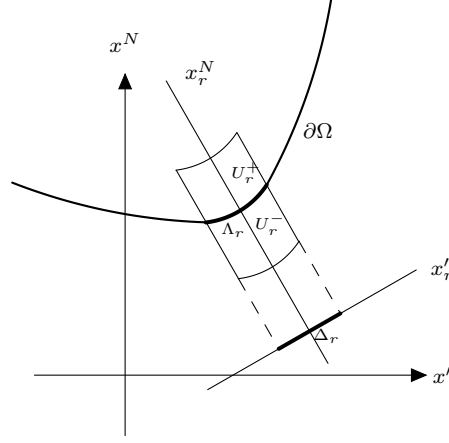
We remind same inclusion relations among Zygmund spaces. If  $1 \leq r < p \leq +\infty$  and  $-\infty < \alpha, \beta < +\infty$ , then we get

$$L^p(\log L)^\alpha(\Omega, \gamma) \subseteq L^r(\log L)^\beta(\Omega, \gamma) .$$

It is clear from definition (2.1) that the space  $L^p(\log L)^\alpha(\Omega, \gamma)$  decreases as  $\alpha$  increases. For more properties we refer to [5].

### 2.3 Smoothness assumptions on the domain

In this paper we deal with integrals involving the values of a  $W^{1,p}$ -function on  $\partial\Omega$ . To this aim we need to have a suitable local description of the set  $\Omega$  and  $\partial\Omega$  is a finite union of graphs. More precisely we will consider smooth domain  $\Omega$  which verifies the following condition (cfr. Chapter 6 of [18] for bounded domain).



**Condition 2.1** Let  $\Omega$  be a domain such that there exist

- i)  $m \in \mathbb{N}$  coordinate systems  $X_r = (x'_r, x_r^N)$  where  $x'_r = (x_r^1, \dots, x_r^{N-1})$  for  $r = 1, 2, \dots, m$ ;
- ii)  $a_i, b_i \in \mathbb{R} \cup \{\pm\infty\}$  for  $i = 1, \dots, N-1$  and  $m$  Lipschitz functions  $a_r$  in  $\overline{\Delta_r} = \{x'_r : x_r^i \in (a_i, b_i) \text{ for } i = 1, \dots, N-1\}$  for  $r = 1, \dots, m$ ;
- iii) a number  $\beta > 0$  such that the sets

$$\Lambda_r = \{(x'_r, x_r^N) \in \mathbb{R}^N : x'_r \in \Delta_r \text{ and } x_r^N = a_r(x'_r)\}$$

are subsets of  $\partial\Omega$ ,  $\partial\Omega = \bigcup_{r=1}^m \Lambda_r$  and the sets

$$U_r^+ = \{(x'_r, x_r^N) \in \mathbb{R}^N : x'_r \in \Delta_r \text{ and } a_r(x'_r) < x_r^N < a_r(x'_r) + \beta\}$$

$$U_r^- = \{(x'_r, x_r^N) \in \mathbb{R}^N : x'_r \in \Delta_r \text{ and } a_r(x'_r) - \beta < x_r^N < a_r(x'_r)\}$$

are subset of  $\Omega$  (after a suitable orthonormal transformation of coordinates).

We observe that the set  $U_r = U_r^+ \cup U_r^-$  is an open subset of  $\mathbb{R}^N$  and there exists an open set  $U_0 \subseteq \overline{U_0} \subset \Omega$  such that the collection  $\{U_r\}_{r=0}^m$  is a open cover of  $\Omega$ . Moreover the collection  $\{U_r\}_{r=1}^m$  is a open cover of  $\partial\Omega$ .

### 3 Sobolev logarithmic inequalities in $W^{1,p}(\Omega, \gamma)$

In this section we prove continuity and compactness of imbedding of  $W^{1,p}(\Omega, \gamma)$  into  $L^p(\log L)^{\frac{1}{2}}(\Omega, \gamma)$ . We will deal also with the case  $p = +\infty$ . The first step is to obtain the analogue of (1.2) and (1.3) when  $u \in W^{1,p}(\Omega, \gamma)$  for  $1 \leq p \leq +\infty$ .

**Proposition 3.1** (*Continuity*) *If  $u \in W^{1,p}(\Omega, \gamma)$  for  $1 \leq p < +\infty$  and  $\Omega$  satisfies condition 2.1, then there exists a positive constant  $C$  depending only on  $p$  and  $\Omega$  such that*

$$\|u\|_{L^p(\log L)^{\frac{1}{2}}(\Omega, \gamma)} \leq C \|u\|_{W^{1,p}(\Omega, \gamma)}, \quad (3.1)$$

*i.e. the embedding of weighted Sobolev space  $W^{1,p}(\Omega, \gamma)$  into the weighted Zygmund space  $L^p(\log L)^{\frac{1}{2}}(\Omega, \gamma)$  is continuous for  $1 \leq p < +\infty$ .*

To prove Proposition 3.1 we need an extension operator  $P$  from  $W^{1,p}(\Omega, \gamma)$  into  $W^{1,p}(\mathbb{R}^N, \gamma)$ . When  $u \in W_0^{1,p}(\Omega, \gamma)$  the natural extension by zero outside  $\Omega$  is continuous without any assumptions on the regularity of the boundary. Working with the space  $W^{1,p}(\Omega, \gamma)$  the situation is more delicate and the regularity of the boundary of  $\Omega$  plays a crucial role.

Using classical tools (see e.g. [4]) it is possible to prove the existence of an extension operator  $P$  from  $W^{1,p}(\Omega, \gamma)$  into  $W^{1,p}(\mathbb{R}^N, \gamma)$  which is linear and continuous. The extension operator allows us to prove the density (for the classical case see e.g. [4]) of  $C^\infty(\overline{\Omega})$  in  $W^{1,p}(\Omega, \gamma)$ .

**Proof of Proposition 3.1.** We consider the extension operator  $P$  from  $W^{1,p}(\Omega, \gamma)$  into  $W^{1,p}(\mathbb{R}^N, \gamma)$  and using (1.2) we obtain for some constant  $c$

$$\begin{aligned} \|u\|_{L^p(\log L)^{\frac{1}{2}}(\Omega, \gamma)} &\leq c \|Pu\|_{L^p(\log L)^{\frac{1}{2}}(\mathbb{R}^N, \gamma)} \leq c \|\nabla(Pu)\|_{L^p(\mathbb{R}^N, \gamma)} \\ &\leq c \|Pu\|_{W^{1,p}(\mathbb{R}^N, \gamma)} \leq c \|u\|_{W^{1,p}(\Omega, \gamma)} \end{aligned}$$

for  $u \in W^{1,p}(\Omega, \gamma)$ . □

**Remark 3.1** The space  $L^p(\log L)^{\frac{1}{2}}(\Omega, \gamma)$  obtained in the Proposition 3.1 is sharp in the class of the Zygmund spaces as the following example shows. We consider  $\Omega = \{x \in \mathbb{R}^N : x_N < \omega\}$  with  $\omega \in \mathbb{R}$  and the function  $u_\delta(x) = \Phi^\delta(x_N)$  with  $-\frac{1}{p} < \delta < 0$ . We observe that  $u_\delta^*(s) = s^\delta$ . By (2.2) we have that

$$\|u_\delta\|_{W^{1,p}(\Omega, \gamma)}^p < +\infty \iff \int_0^{\gamma(\Omega)} s^{\delta p} (1 - \log s)^{\frac{p}{2}} ds < +\infty,$$

this means that  $u_\delta \in W^{1,p}(\Omega, \gamma)$  and

$$\|u_\delta\|_{L^p(\log L)^\alpha(\Omega, \gamma)} < +\infty \iff \alpha \leq \frac{1}{2}.$$

**Remark 3.2** By Proposition 3.1 follows the continuity of the embedding of Sobolev space  $W^{m,p}(\Omega, \gamma)$   $m \geq 1$  into the Zygmund space  $L^p(\log L)^{m\alpha}(\Omega, \gamma)$  for  $\alpha \leq \frac{1}{2}$ . A similar result for  $\Omega = \mathbb{R}^N$  is proved in [15].

Let now consider Lipschitz continuous functions.

**Proposition 3.2** *If  $u$  is a Lipschitz continuous function,  $\Omega$  satisfies condition 2.1 and  $\lim_{x \in \Omega, |x| \rightarrow +\infty} u(x) = 0$ , then there exists a positive constant  $C$  depending only on  $\Omega$  such that*

$$\|u\|_{L^\infty(\log L)^{-\frac{1}{2}}(\Omega, \gamma)} \leq C \left( \|\nabla u\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)} \right). \quad (3.2)$$

**Remark 3.3** The space  $L^\infty(\log L)^{-\frac{1}{2}}(\Omega, \gamma)$  obtained in the Proposition 3.2 is sharp in the class of the Zygmund spaces as the following example shows. We consider  $\Omega = \{x \in \mathbb{R}^N : x_N < \omega\}$  with  $\omega \in \mathbb{R}$  and the function  $u_\delta(x) = (1 - \log \Phi(x_N))^\delta$  with  $0 < \delta \leq \frac{1}{2}$ . We observe that  $u_\delta^\oplus(s) = (1 - \log s)^\delta$ . By (2.2) we have that

$$\|\nabla u\|_{L^\infty(\Omega)} < +\infty \iff \sup_{s \in (0, \gamma(\Omega))} (1 - \log s)^{\delta - \frac{1}{2}} < +\infty,$$

this means that  $u_\delta$  is a Lipschitz continuous function and

$$\|u_\delta\|_{L^\infty(\log L)^\alpha(\Omega, \gamma)} < +\infty \iff \alpha \leq -\frac{1}{2}.$$

In order to prove Proposition 3.2 we can argue as in the proof of Proposition 3.1: we need the extension operator  $P$  and the inequality (1.3). Let us observe that the boundary conditions  $\lim_{x \in \Omega, |x| \rightarrow +\infty} u(x) = 0$  and  $u|_{\partial\Omega} = 0$  are necessary to obtain the Polya-Szëgo inequality for  $p = +\infty$ , that is a crucial tool to prove (1.3) and (3.2).

**Proposition 3.3** (Compactness) *Let  $1 \leq p < +\infty$  and let  $\Omega$  satisfy condition 2.1. Then the embedding of  $W^{1,p}(\Omega, \gamma)$  into  $L^p \log L^\beta(\Omega, \gamma)$  is compact if  $\beta < \frac{1}{2}$ .*

**Proof.** It is enough to prove the compactness of the embedding of  $W^{1,p}(\Omega, \gamma)$  into  $L^1(\Omega, \gamma)$ . Indeed we have that any bounded set of  $L^p \log L^{\frac{1}{2}}(\Omega, \gamma)$  which is precompact in  $L^1(\Omega, \gamma)$  is also precompact in  $L^p \log L^\beta(\Omega, \gamma)$  with  $\beta < \frac{1}{2}$  (see e.g. Theorem 8.23 of [2]).

Let  $S$  be a bounded set in  $W^{1,p}(\Omega, \gamma)$ , then  $S$  is bounded in  $L^1(\Omega, \gamma)$  too. Using a characterization of precompact sets of Lebesgue spaces (see e.g. Theorem 2.21 of [2]) we have to prove that for any number  $\varepsilon > 0$  there exists a number  $\delta > 0$  and a subset  $G \subset\subset \Omega$  such that for any  $u \in S$  and every  $h \in \mathbb{R}^N$  with  $|h| < \delta$  the following conditions hold:

$$a) \int_{\Omega} |\tilde{u}(x+h)\varphi(x+h) - \tilde{u}(x)\varphi(x)| dx < \varepsilon \quad (3.3)$$

$$b) \int_{\Omega - \bar{G}} |u| d\gamma < \varepsilon, \quad (3.4)$$

where  $\tilde{u}$  is the zero extension of  $u$  outside  $\Omega$ .

Let  $\varepsilon > 0$  and  $\Omega_j = \left\{ x \in \Omega : \text{dist}(x, \partial\Omega) > \frac{1}{j} \right\}$  for  $j \in \mathbb{N}$ . By (3.1) we have for some constant  $c$

$$\begin{aligned} \int_{\Omega - \Omega_j} |u| d\gamma &\leq \left( \int_0^{\gamma(\Omega - \Omega_j)} \left[ (1 - \log t)^{\frac{1}{2}} u^*(t) \right]^p dt \right)^{\frac{1}{p}} \left( \int_{\Omega - \Omega_j} (1 - \log t)^{-\frac{p'}{2}} \right)^{\frac{1}{p'}} \\ &\leq c \|u\|_{W^{1,p}(\Omega, \gamma)} \left( \int_{\Omega - \Omega_j} (1 - \log t)^{-\frac{p'}{2}} \right)^{\frac{1}{p'}} \end{aligned}$$

Since the Gauss measure of  $\Omega$  is finite, we can choose  $j$  big enough to have

$$\int_{\Omega - \Omega_j} |u| d\gamma < \varepsilon, \quad (3.5)$$

(i.e. (3.4) holds) and for  $h \in \mathbb{R}^N$

$$\int_{\Omega - \Omega_j} |\tilde{u}(x+h)\varphi(x+h) - \tilde{u}(x)\varphi(x)| dx < \frac{\varepsilon}{2}. \quad (3.6)$$



Let  $|h| \leq \frac{1}{j}$ , then  $x + th \in \Omega_{2j}$  if  $x \in \Omega$  and  $t \in [0, 1]$ . Let  $u \in C^\infty(\overline{\Omega})$ , we have for some constant  $c$

$$\begin{aligned}
& \int_{\Omega_j} |\tilde{u}(x+h)\varphi(x+h) - \tilde{u}(x)\varphi(x)| dx \\
& \leq \int_{\Omega_j} \int_0^1 \left| \frac{d}{dt} \tilde{u}(x+th)\varphi(x+th) \right| dt dx \text{ (togliere)} \\
& \leq \int_{\Omega_j} \int_0^1 |\nabla \tilde{u}(x+th)h\varphi(x+th) - \tilde{u}(x+th)\varphi(x+th)(x+th)h| dt dx \\
& \leq |h| \left( \int_{\Omega_{2j}} |\nabla \tilde{u}(y)\varphi(y)| dy + \int_{\Omega_{2j}} |\tilde{u}(y)\varphi(y)y| dy \right) \\
& \leq c|h| \left( \|\nabla u\|_{L^p(\Omega, \gamma)}^p + \|u\|_{L^p(\log L)^{\frac{1}{2}}(\Omega, \gamma)}^p \right) \leq c|h| \|u\|_{W^{1,p}(\Omega, \gamma)}^p.
\end{aligned} \tag{3.7}$$

In the last inequalities we have used (3.1) and the fact that  $f(x) = |x| \in L^{p'}(\log L)^{-\frac{1}{2}}(\Omega, \gamma)$ . Indeed since  $\gamma_f(t) = 1 - \gamma(B(0, t))$ , one can easily check that

$$\int_0^{\gamma(\Omega)} (1 - \log s)^{-\frac{p'}{2}} [(|x|)^\otimes(s)]^{p'} ds = \int_0^{+\infty} t^{p'} (1 - \log \gamma_f(t))^{-\frac{p'}{2}} \gamma'_f(t) dt < +\infty.$$

Because of the density of  $C^\infty(\overline{\Omega})$  in  $W^{1,p}(\Omega, \gamma)$ , (3.7) holds for every  $u$  in  $W^{1,p}(\Omega, \gamma)$  and then for  $|h|$  small enough by (3.6) and (3.7) we obtain (3.3)  $\square$

**Remark 3.4** Obviously the compactness results holds for  $W_0^{1,p}(\Omega, \gamma)$  for any domain  $\Omega$ .

**Remark 3.5** The compactness proved in Proposition 3.3 implies the compact embedding of Sobolev space  $W^{m,p}(\Omega, \gamma)$   $m \geq 1$  into the Zygmund space  $L^p(\log L)^{m\beta}(\Omega, \gamma)$  for  $\beta < \frac{1}{2}$ .

The compactness can be used to obtain a Poincaré-Wirtinger type inequality.

**Proposition 3.4** *Let  $\Omega$  be a connected domain satisfying condition 2.1. Assume  $1 \leq p < +\infty$ . Then there exists a positive constant  $C$ , depending only on  $p$  and  $\Omega$ , such that*

$$\|u - u_\Omega\|_{L^p(\Omega, \gamma)} \leq C \|\nabla u\|_{L^p(\Omega, \gamma)} \tag{3.8}$$

for any  $u \in W^{1,p}(\Omega, \gamma)$ , where  $u_\Omega = \frac{1}{\gamma(\Omega)} \int_\Omega u d\gamma$ .

**Proof.** We precede as in the classical case. We argue by contradiction, then there would exist for any  $k \in \mathbb{N}$  a function  $u_k \in W^{1,p}(\Omega, \gamma)$  such that

$$\|u_k - (u_k)_\Omega\|_{L^p(\Omega, \gamma)} > k \|\nabla u_k\|_{L^p(\Omega, \gamma)}.$$

We renormalize by defining

$$v_k = \frac{u_k - (u_k)_\Omega}{\|u_k - (u_k)_\Omega\|_{L^p(\Omega, \gamma)}}. \quad (3.9)$$

Then

$$(v_k)_\Omega = 0, \|v_k\|_{L^p(\Omega, \gamma)} = 1$$

and

$$\text{and } \|\nabla v_k\|_{L^p(\Omega, \gamma)} < \frac{1}{k}. \quad (3.10)$$

In particular the functions  $\{v_k\}_{k \in \mathbb{N}}$  are bounded in  $W^{1,p}(\Omega, \gamma)$ . Then by the previous theorem there exists a subsequence still denoted by  $\{v_k\}_{k \in \mathbb{N}}$  and a function  $v$  such that

$$v_k \rightarrow v \quad \text{in } L^p(\Omega, \gamma).$$

Moreover by (3.9) it follows that

$$v_\Omega = 0 \text{ and } \|v\|_{L^p(\Omega, \gamma)} = 1. \quad (3.11)$$

On the other hand, (3.10) implies for any  $\psi \in C_0^\infty(\Omega)$  and  $i = 1, \dots, N$

$$\begin{aligned} \int_\Omega v \frac{\partial \psi}{\partial x_i} \varphi dx - \int_\Omega v \psi x_i \varphi dx &= \lim_{k \rightarrow +\infty} \left( \int_\Omega v_k \frac{\partial \psi}{\partial x_i} \varphi dx - \int_\Omega v_k \psi x_i \varphi dx \right) \\ &= \lim_{k \rightarrow +\infty} - \int_\Omega \frac{\partial v_k}{\partial x_i} \psi \varphi dx = 0. \end{aligned}$$

Consequently  $v \in W^{1,p}(\Omega, \gamma)$  and  $\nabla v = 0$  a.e. Then  $v$  is constant since  $\Omega$  is connected. In particular by the first estimate in (3.11) we must have  $v \equiv 0$ ; in which case  $\|v\|_{L^p(\Omega, \gamma)} = 0$ . This contradiction establishes the estimate (3.8). □

**Remark 3.6** The previous proof works in a more general case. Let  $\Omega$  be a connected domain satisfying condition 2.1 and let  $V \subset W^{1,p}(\Omega, \gamma)$  be a linear subspace of  $W^{1,p}(\Omega, \gamma)$  with  $1 \leq p < +\infty$  which is closed and such that the only constant function belonging to  $V$  is the function which is identically

zero. Then there exists a positive constant  $C$ , depending only on  $p$  and  $\Omega$ , such that

$$\|v\|_{L^p(\Omega, \gamma)} \leq C \left( \int_{\Omega} \sum_{i=1}^N \left| \frac{\partial v}{\partial x_i} \right|^p d\gamma \right)^{\frac{1}{p}} \quad \forall v \in V.$$

**Remark 3.7** (*Application to PDE*) Let  $\Omega$  be a connected domain satisfying condition 2.1. Let us consider the semicoercive homogeneous Neumann problem

$$\begin{cases} -(u_{x_i} \varphi)_{x_i} = f \varphi & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.12)$$

where  $f \in L^2(\log L)^{-\frac{1}{2}}(\Omega, \gamma)$  and  $\nu$  is the external normal. Using classical tools (see e.g. [4] Theorem 6.2.3) and inequalities (3.1) and (3.8) it follows that problem (3.12) has a weak solution in  $W^{1,2}(\Omega, \gamma)$  if and only if  $\int_{\Omega} f d\gamma = 0$ . In particular there exists a unique weak solution in  $X = \{u \in W^{1,2}(\Omega, \gamma) : \int_{\Omega} u d\gamma = 0\}$  by Lax-Milgram theorem.

We consider also the following eigenvalue problem related to the equation of quantum harmonic oscillator

$$\begin{cases} -(u_{x_i} \varphi)_{x_i} = \lambda u & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.13)$$

Arguing in a classical way (see e.g. [4] Theorem 8.6.1), using inequality (3.8) and the compactness of the embedding from  $W^{1,2}(\Omega, \gamma)$  into  $L^2(\Omega, \gamma)$ , it follows that there exists an increasing sequence of eigenvalues of the problem (3.13) which tends to infinity and a Hilbertian basis of eigenfunctions in  $L^2(\Omega, \gamma)$ . Moreover for  $\lambda_1 = 0$ , the corresponding eigenfunction  $u_1 = \text{const} \neq 0$  and the first nontrivial eigenvalue  $\lambda_2$  has the following characterization

$$\lambda_2 = \min \left\{ \frac{\|\nabla u\|_{L^2(\Omega, \gamma)}}{\|u\|_{L^2(\Omega, \gamma)}}, u \in W^{1,2}(\Omega, \gamma) : \int_{\Omega} u d\gamma = 0 \right\}.$$

## 4 Sobolev logarithmic trace inequalities

In this section we deal with integrals involving the values of a  $C^\infty$ -function on  $\partial\Omega$ . We prove that a certain integral of the function on  $\partial\Omega$  is bounded by the  $W^{1,p}$ -norm on  $\Omega$ . This inequality will be crucial to define trace operator (see §5).

**Proposition 4.1** *Let  $\Omega$  be a domain satisfying condition 2.1 and  $1 \leq p < +\infty$ . For every  $u \in C^\infty(\overline{\Omega})$  there exists a positive constant  $C$  depending only on  $p$  and  $\Omega$  such that*

$$\int_{\partial\Omega} |u|^p \log^{\frac{p-1}{2}}(2 + |u|) \varphi dS \leq C \|u\|_{W^{1,p}(\Omega,\gamma)}^p \quad (4.1)$$

**Remark 4.1** We obtain the same result if we replace the first member of (4.1) with the quantity  $\int_{\partial\Omega} u^p (\log^+(|u|))^{\frac{p-1}{2}} \varphi dS$ .

**Proof.** Following classical tools (see Chapter 6 of [18] ) it is enough to prove the existence of a constant  $C_T > 0$  such that for any function  $u \in C^\infty(\overline{\Omega})$  whose supports is in  $\Lambda_r \cup U_r^+$  we have (4.1). After suitable transformation that maps  $\Delta_r \times ]0, \beta[$  onto  $U_r^+$  and  $\Delta_r \times \{0\}$  onto  $\Lambda_r$ , we can reduce to consider  $u$  such that the support is in  $\Delta_r \times [0, \beta[$ . Then it is sufficient to prove the existence of a constant  $C > 0$  such that for any function  $u \in C^\infty(\overline{\Delta_r} \times [0, \beta[)$  whose supports is in  $\Delta_r \times [0, \beta[$

$$\int_{\Delta_r} |u(x'_r, 0)|^p \log^{\frac{p-1}{2}}(2 + |u(x'_r, 0)|) \varphi(x'_r, 0) dx'_r \leq C \|u\|_{W^{1,p}(\Delta_r \times ]0, \beta[, \gamma)}^p \cdot \quad (4.2)$$

holds. In (4.2) we have denoted by  $u$  the composition of  $u$  with the change of coordinates.

Now we prove (4.2). For some constant  $c$  that can varies from line to line we have

$$\begin{aligned} & \int_{\Delta_r} |u(x'_r, 0)|^p \log^{\frac{p-1}{2}}(2 + |u(x'_r, 0)|) \varphi(x'_r, 0) dx'_r \\ & \leq c(A_1 + A_2 + A_3) \end{aligned} \quad (4.3)$$

where

$$\begin{aligned} A_1 &= \int_{\Delta_r} \int_{\beta}^0 p |u(x'_r, x_r^N)|^{p-1} \log^{\frac{p-1}{2}}(2 + |u(x'_r, x_r^N)|) \left| \frac{\partial u}{\partial x_r^N}(x'_r, x_r^N) \right| \varphi(x'_r, x_r^N) dx_r^N dx'_r \\ A_2 &= \int_{\Delta_r} \int_{\beta}^0 \frac{p-1}{2} |u(x'_r, x_r^N)|^p \frac{\log^{\frac{p-1}{2}-1}(2 + |u(x'_r, x_r^N)|)}{2 + |u(x'_r, x_r^N)|} \left| \frac{\partial u}{\partial x_r^N}(x'_r, x_r^N) \right| \varphi(x'_r, x_r^N) dx_r^N dx'_r \\ A_3 &= \int_{\Delta_r} \int_{\beta}^0 |u(x'_r, x_r^N)|^p \log^{\frac{p-1}{2}}(2 + |u(x'_r, x_r^N)|) \varphi(x'_r, x_r^N) |x_r^N| dx_r^N dx'_r. \end{aligned}$$

We observe that the function  $f(x) = x_r^N \in L^\infty(\log L)^{-\frac{1}{2}}(\Delta_r \times ]0, \beta[, \gamma)$ .  
Indeed  $\gamma_f(t) = 2\Phi(-t)$  and using (2.2) we have

$$\begin{aligned} \sup_{t \in (0, \gamma(\Delta_r \times ]0, \beta[))} (1 - \log t)^{-\frac{1}{2}} f^\oplus(t) &= \sup_{t \in (0, \gamma(\Delta_r \times ]0, \beta[))} (1 - \log t)^{-\frac{1}{2}} \left( -\Phi^{-1} \left( \frac{t}{2} \right) \right) \\ &\leq c \sup_{t \in (0, \gamma(\Delta_r \times ]0, \beta[))} (1 - \log t)^{-\frac{1}{2}} (2 \log \frac{2}{t})^{\frac{1}{2}} < +\infty. \end{aligned}$$

Then we obtain

$$A_3 \leq c \|u\|_{L^p(\log L)^{\frac{1}{2p'}}(\Delta_r \times ]0, \beta[, \gamma)}^p \|x_r^N\|_{L^\infty(\log L)^{-\frac{1}{2}}(\Delta_r \times ]0, \beta[, \gamma)} \quad (4.4)$$

Moreover using Hölder inequality, we obtain

$$A_1 \leq c \left( \int_{\Delta_r \times ]0, \beta[} |u(x'_r, x_r^N)|^p \log^{\frac{p}{2}}(2 + |u(x'_r, x_r^N)|) \varphi(x'_r, x_r^N) dx_r^N dx'_r \right)^{\frac{1}{p}} \quad (4.5)$$

$$\times \left( \int_{\Omega} \left| \frac{\partial u}{\partial x_r^N}(x'_r, x_r^N) \right|^p \varphi(x'_r, x_r^N) dx_r^N dx'_r \right)^{\frac{1}{p}}$$

$$A_2 \leq c \left( \int_{\Delta_r \times ]0, \beta[} |u(x'_r, x_r^N)|^p \log^{(\frac{p-1}{2}-1)p'}(2 + |u(x'_r, x_r^N)|) \varphi(x'_r, x_r^N) dx_r^N dx'_r \right)^{\frac{1}{p}} \times \quad (4.6)$$

$$\times \left( \int_{\Omega} \left| \frac{\partial u}{\partial x_r^N}(x'_r, x_r^N) \right|^p \varphi(x'_r, x_r^N) dx_r^N dx'_r \right)^{\frac{1}{p}}$$

We observe that

$$\begin{aligned} &\int_{\Delta_r \times ]0, \beta[} |u(x'_r, x_r^N)|^p \log^{(\frac{p-1}{2}-1)p'}(2 + |u(x'_r, x_r^N)|) \varphi(x'_r, x_r^N) dx_r^N dx'_r \\ &\leq c \int_{\Delta_r \times ]0, \beta[} |u(x'_r, x_r^N)|^p \log^{\frac{p}{2}}(2 + |u(x'_r, x_r^N)|) \varphi(x'_r, x_r^N) dx_r^N dx'_r \end{aligned}$$

and

$$\left( \int_{\Delta_r \times ]0, \beta[} |u(x'_r, x_r^N)|^p \log^{\frac{p}{2}}(2 + |u(x'_r, x_r^N)|) \varphi(x'_r, x_r^N) dx_r^N dx'_r \right) \leq \quad (4.7)$$

$$\begin{aligned}
&= \int_0^{\gamma(\Delta_r \times ]0, \beta[)} \left[ u^{\otimes}(t) \log^{\frac{1}{2}}(2 + u^{\otimes}(t)) \right]^p dt \\
&\leq c \left( \int_0^{\gamma(\Delta_r \times ]0, \beta[)} \left[ (1 - \log t)^{\frac{1}{2}} u^{\otimes}(t) \right]^p dt \right),
\end{aligned}$$

because  $\log(2 + u^{\otimes}(t))$  is dominated by a multiple of  $(1 - \log t)$ . Indeed  $L^p(\log L)^{\frac{1}{2}} \subset L^p \subset L^{p, \infty}$ , then  $u^{\otimes}(t) \leq ct^{-\frac{1}{p}}$  for some positive constant.

Putting (4.4)-(4.7) in (4.3) and using Proposition 3.1 we have

$$\begin{aligned}
&\int_{\Delta_r} |u(x'_r, 0)|^p \log^{\frac{p-1}{2}}(2 + |u(x'_r, 0)|) \varphi(x'_r, 0) dx'_r \\
&\leq c \|u\|_{L^p(\log L)^{\frac{1}{2}}(\Delta_r \times ]0, \beta[, \gamma)}^{p-1} \|\nabla u\|_{L^p(\Delta_r \times ]0, \beta[, \gamma)} + c \|u\|_{L^p(\log L)^{\frac{1}{2p'}}(\Delta_r \times ]0, \beta[, \gamma)}^p \|x_r^N\|_{L^\infty(\log L)^{-\frac{1}{2}}(\Delta_r \times ]0, \beta[, \gamma)} \\
&\leq c \|u\|_{W^{1,p}(\Delta_r \times ]0, \beta[, \gamma)}^{p-1} \|\nabla u\|_{L^p(\Delta_r \times ]0, \beta[, \gamma)} + c \|u\|_{W^{1,p}(\Delta_r \times ]0, \beta[, \gamma)}^p \\
&\leq c \|u\|_{W^{1,p}(\Delta_r \times ]0, \beta[, \gamma)}^p.
\end{aligned}$$

□

**Remark 4.2** In (4.1) the exponent  $\frac{p-1}{2}$  of the logarithmic is sharp as the following example shows. We consider  $\Omega = \{x \in \mathbb{R}^N : x_N < \omega\}$  with  $\omega \in \mathbb{R}$  and the function  $u_\delta(x) = \Phi^\delta(x_N)$  with  $-\frac{1}{p} < \delta < 0$  as in Remark 3.1. We have that

$$\int_{\partial\Omega} |u|^p \log^\beta(2 + |u|) \varphi dS = A_1 + A_2 + A_3$$

where

$$\begin{aligned}
A_1 &= \int \int_{\Omega} p |u(x', x_N)|^{p-1} \log^\beta(2 + |u(x', x_N)|) \left| \frac{\partial u}{\partial x_N}(x', x_N) \right| \varphi(x', x_N) dx' dx_N \\
A_2 &= \int \int_{\Omega} \frac{p-1}{2} |u(x', x_N)|^p \frac{\log^{\beta-1}(2 + |u(x', x_N)|)}{2 + |u(x', x_N)|} \left| \frac{\partial u}{\partial x_N}(x', x_N) \right| \varphi(x', x_N) dx' dx_N \\
A_3 &= - \int \int_{\Omega} |u(x', x_N)|^p \log^{\frac{p-1}{2}}(2 + |u(x', x_N)|) \varphi(x', x_N) x_N dx' dx_N.
\end{aligned}$$

By (2.2) we have that

$$A_1 < +\infty \iff \beta \leq \frac{p-1}{2} < \frac{p}{2}.$$

Moreover  $A_2 \leq cA_1$  and

$$A_3 < +\infty \iff \beta \leq \frac{p}{2},$$

because  $x_N \in L^\infty(\log L)^{-\frac{1}{2}}(\Omega, \gamma)$ .

If  $p = +\infty$  we can prove the following result.

**Proposition 4.2** *Let  $\Omega$  be a domain satisfying condition 2.1. For every  $u \in C^\infty(\overline{\Omega})$  such that  $\lim_{x \in \Omega, |x| \rightarrow +\infty} u = 0$  and every  $\lambda \in (0, 1)$ , there exists a positive constant depending on  $\Omega$  and  $\lambda$  such that*

$$\int_{\partial\Omega} \exp\left(\lambda |u|^2\right) \varphi \, dS \leq C \exp\left[\left(\|\nabla u\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)}\right)^2\right] \quad (4.8)$$

$$\left(\|\nabla u\|_{L^\infty(\Omega)} \left(\|\nabla u\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)}\right) + 1\right).$$

**Proof of Proposition 4.2.** As in the proof of Proposition 4.1 it is sufficient to prove for any functions  $u \in C^\infty(\overline{\Delta_r} \times [0, \beta])$  whose supports is in  $\Delta_r \times [0, \beta]$  and any  $\lambda \in (0, 1)$  the following inequality

$$\int_{\Delta_r} \exp\left(\lambda |u(x'_r, 0)|^2\right) \varphi(x'_r, 0) dx'_r \leq C \exp\left[\left(\|\nabla u\|_{L^\infty(\overline{\Delta_r} \times [0, \beta])} + \|u\|_{L^\infty(\overline{\Delta_r} \times [0, \beta])}\right)^2\right] \times \quad (4.9)$$

$$\times \left(\|\nabla u\|_{L^\infty(\overline{\Delta_r} \times [0, \beta])} \left(\|\nabla u\|_{L^\infty(\overline{\Delta_r} \times [0, \beta])} + \|u\|_{L^\infty(\overline{\Delta_r} \times [0, \beta])}\right) + 1\right)$$

holds for some positive constant  $C$  not depending on  $u$ .

Now we prove (4.9). For some constant  $c$  that can varies from line to line we have

$$\int_{\Delta_r} \exp\left(\lambda |u(x'_r, 0)|^2\right) \varphi(x'_r, 0) dx'_r \leq c(B_1 + B_2) \quad (4.10)$$

where

$$B_1 = \int_{\Delta_r} \int_{\beta}^0 \lambda |u(x'_r, 0)| \exp\left(\lambda |u(x'_r, 0)|^2\right) \left| \frac{\partial u}{\partial x_r^N}(x'_r, x_r^N) \right| \varphi(x'_r, x_r^N) \, dx_r^N \, dx'_r$$

$$B_2 = \int_{\Delta_r} \int_{\beta}^0 \exp\left(\lambda |u(x'_r, 0)|^2\right) \varphi(x'_r, x_r^N) |x_r^N| \, dx_r^N \, dx'_r.$$

Moreover, since  $u^{\otimes}(t) \leq \|u\|_{L^\infty(\log L)^{-\frac{1}{2}}(\overline{\Delta_r} \times [0, \beta])} (1 - \log t)^{\frac{1}{2}}$  in  $\overline{\Delta_r} \times [0, \beta]$ , we have

$$\begin{aligned}
B_1 &\leq \lambda \|\nabla u\|_{L^\infty(\overline{\Delta_r} \times [0, \beta])} \int_{\Delta_r} \int_{\beta}^0 |u(x'_r, 0)| \exp\left(\lambda |u(x'_r, 0)|^2\right) \varphi(x'_r, x_r^N) dx_r^N dx'_r \\
&= \lambda \|\nabla u\|_{L^\infty(\overline{\Delta_r} \times [0, \beta])} \int_0^{\gamma(\overline{\Delta_r} \times [0, \beta])} |u^{\otimes}(t)| \exp\left(\lambda |u^{\otimes}(t)|^2\right) dt \\
&\leq \lambda \exp\left(\|u\|_{L^\infty(\log L)^{-\frac{1}{2}}(\overline{\Delta_r} \times [0, \beta])}^2\right) \|\nabla u\|_{L^\infty(\overline{\Delta_r} \times [0, \beta])} \|u\|_{L^\infty(\log L)^{-\frac{1}{2}}(\overline{\Delta_r} \times [0, \beta])} \times \\
&\quad \times \int_0^{\gamma(\overline{\Delta_r} \times [0, \beta])} \frac{(1 - \log t)^{\frac{1}{2}}}{t^\lambda} dt.
\end{aligned} \tag{4.11}$$

and

$$B_2 \leq \|x_r^N\|_{L^\infty(\log L)^{-\frac{1}{2}}(\Delta_r \times ]0, \beta[, \gamma)} \exp\left(\|u\|_{L^\infty(\log L)^{-\frac{1}{2}}(\overline{\Delta_r} \times [0, \beta])}^2\right) \int_0^{\gamma(\overline{\Delta_r} \times [0, \beta])} \frac{(1 - \log t)^{\frac{1}{2}}}{t^\lambda} dt. \tag{4.12}$$

For any  $\lambda \in (0, 1)$  the integrals in (4.11) and (4.12) are finite and

$$B_1 \leq c \|\nabla u\|_{L^\infty(\overline{\Delta_r} \times [0, \beta])} \|u\|_{L^\infty(\log L)^{-\frac{1}{2}}(\overline{\Delta_r} \times [0, \beta])} \exp\left(\|u\|_{L^\infty(\log L)^{-\frac{1}{2}}(\overline{\Delta_r} \times [0, \beta])}^2\right) \tag{4.13}$$

and

$$B_2 \leq c \exp\left(\|u\|_{L^\infty(\log L)^{-\frac{1}{2}}(\overline{\Delta_r} \times [0, \beta])}^2\right) \tag{4.14}$$

for some constant  $c$  depending on  $\lambda$  and  $\overline{\Delta_r} \times [0, \beta]$ .

Putting (4.13) and (4.14) in (4.10) and using (3.2) we obtain (4.9). □

**Remark 4.3** In (4.8) the power 2 in the argument of the exponential is sharp as the following example shows. In order to show that we need to consider  $\Omega = \{x \in \mathbb{R}^N : x_N < \omega\}$  with  $\omega \in \mathbb{R}$  and the function  $u_\delta(x) = u_\delta(x) = (1 - \log \Phi(x_N))^\delta$  with  $0 < \delta \leq \frac{1}{2}$  as in Remark 3.3 and argue as in Remark 4.2.

## 5 Trace operator

In this section the "boundary values" or trace of functions in Sobolev spaces are studied.



If  $\Omega$  is a domain satisfying condition 2.1, given a smooth function  $u \in C^\infty(\overline{\Omega}) \subset W^{1,p}(\Omega, \gamma)$  we can define the restriction to the boundary  $u|_{\partial\Omega}$ . It turn out that this restriction operator can be extended from smooth functions to  $W^{1,p}(\Omega, \gamma)$  giving a linear continuous operator from  $W^{1,p}(\Omega, \gamma)$  to  $L^p(\partial\Omega, \gamma)$ , the space of the measurable functions defined almost everywhere on  $\partial\Omega$  such that

$$\int_{\partial\Omega} |u|^p \varphi \, d\mathcal{H}^{N-1} < +\infty.$$

We stress that  $L^p(\partial\Omega, \gamma)$  is a Banach space with respect to the norm  $\|u\|_{L^p(\partial\Omega, \gamma)} = \left(\int_{\partial\Omega} |u|^p \varphi \, d\mathcal{H}^{N-1}\right)^{\frac{1}{p}}$ .

Using the logarithmic Sobolev inequalities (4.1), there exists a constant  $C > 0$  such that for every  $u \in C^\infty(\overline{\Omega})$

$$\|u\|_{L^p(\partial\Omega, \gamma)} \leq C \|u\|_{W^{1,p}(\Omega, \gamma)}^p. \quad (5.1)$$

It follows that the operator

$$\begin{aligned} T : C^\infty(\overline{\Omega}) &\rightarrow L^p(\partial\Omega, \gamma) \\ u &\rightarrow Tu = u/\partial\Omega \end{aligned}$$

is linear and continuous from  $(C^\infty(\overline{\Omega}), \|\cdot\|_{W^{1,p}(\Omega, \gamma)})$  into  $(L^p(\partial\Omega, \gamma), \|\cdot\|_{L^p(\partial\Omega, \gamma)})$ .

By Hahn-Banach theorem and the density of  $C^\infty(\overline{\Omega})$  in  $W^{1,p}(\Omega, \gamma)$  the operator can be extended to  $W^{1,p}(\Omega, \gamma)$ . This linear continuous operator from  $W^{1,p}(\Omega, \gamma)$  to  $L^p(\partial\Omega, \gamma)$  is called trace operator of  $u$  on  $\partial\Omega$ . Then there exists a constant  $C > 0$  such that

$$\|Tu\|_{L^p(\partial\Omega, \gamma)} \leq C \|u\|_{W^{1,p}(\Omega, \gamma)} \quad \text{for every } u \in W^{1,p}(\Omega, \gamma), \quad (5.2)$$

that implies that  $W^{1,p}(\Omega, \gamma)$  is continuous imbedded in  $L^p(\partial\Omega, \gamma)$ .

Moreover the trace operator is compact for  $1 \leq p < +\infty$ . Indeed let  $\{u_n\}_{n \in \mathbb{N}}$  be a bounded sequence in  $W^{1,p}(\Omega, \gamma)$ , we will prove the existence of a Cauchy subsequence in  $L^p(\partial\Omega, \gamma)$ . By Proposition 3.3, there exists a Cauchy subsequence, still denoted by  $\{u_n\}_{n \in \mathbb{N}}$ , in  $L^p(\log L)^{\frac{1}{2p'}}(\Omega, \gamma)$ . Moreover arguing as in the proof of the inequality (4.1) we have

$$\begin{aligned} \|Tu_n - Tu_m\|_{L^p(\partial\Omega, \gamma)}^p &\leq \int_{\partial\Omega} |Tu_n - Tu_m|^p \log^{\frac{p-1}{2}}(2 + |Tu_n - Tu_m|) \varphi \, d\mathcal{H}^{N-1} \\ &\leq c \|u_n - u_m\|_{L^p(\log L)^{\frac{1}{2}}(\Omega, \gamma)}^{p-1} \|\nabla(u_n - u_m)\|_{L^p(\Omega, \gamma)} \\ &\quad + c \|u_n - u_m\|_{L^p(\log L)^{\frac{p-1}{2}}(\Omega, \gamma)}^p \|x_N\|_{L^\infty(\log L)^{\frac{1}{2}}(\Omega, \gamma)}, \end{aligned}$$

then  $\{u_n\}_{n \in \mathbb{N}}$  is a Cauchy sequence in  $L^p(\partial\Omega, \gamma)$  too.

The norm of the trace operator is given by

$$\inf_{u \in W^{1,p}(\Omega, \gamma) - W_0^{1,p}(\Omega, \gamma)} \frac{\|u\|_{W^{1,p}(\Omega, \gamma)}^p}{\|Tu\|_{L^p(\partial\Omega, \gamma)}^p} \quad (5.3)$$

and this value is the best constant in the trace inequality (5.2). The trace operator is compact, therefore an easy compactness arguments prove that there exist extremals in (5.3). These extremals turn out to be the weak solution of

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2} \nabla u \varphi) = |u|^{p-2} u \varphi & \text{in } \Omega \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda |u|^{p-2} u & \text{on } \partial\Omega, \end{cases} \quad (5.4)$$

where  $\lambda$  is the first nontrivial eigenvalue.

When  $p = 2$  and  $\Omega$  is a connected domain satisfying condition 2.1, using classical tools, compactness of the trace operator from  $W^{1,2}(\Omega, \gamma)$  to  $L^2(\partial\Omega, \gamma)$  and (3.8) it follows that there exists an increasing sequence of eigenvalues of the problem (5.4) which tends to infinity and a Hilbertian basis of eigenfunctions in  $L^2(\Omega, \gamma)$ .

Moreover the continuity of the trace operator from  $W^{1,2}(\Omega, \gamma)$  to  $L^2(\partial\Omega, \gamma)$  and (3.8) allow us to investigate about the existence of a weak solution of the following semicoercive nonhomogeneous Neumann problem

$$\begin{cases} -(u_{x_i} \varphi)_{x_i} = f \varphi & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega, \end{cases}$$

where  $\Omega$  is a connected domain satisfying condition 2.1,  $f \in L^2(\log L)^{\frac{1}{2}}(\Omega, \gamma)$  and  $g \in L^2(\partial\Omega, \gamma)$ . Indeed using classical tools (see e.g. [4] Theorem 6.2.5) we obtain that there exists a weak solution in  $W^{1,2}(\Omega, \gamma)$  if and only if  $\int_{\Omega} f d\gamma + \int_{\partial\Omega} g \varphi dH^{N-1} = 0$ . In particular there exists a unique weak solution in  $X = \{u \in W^{1,2}(\Omega, \gamma) : \int_{\Omega} u d\gamma = 0\}$  by Lax-Milgram theorem.

## 6 Poincaré trace inequality

Arguing as in Proposition 3.4 (see Remark 3.6 too), we prove the following Poincaré type inequality.

**Proposition 6.1** *Let  $\Omega$  be a connected domain satisfying condition 2.1 and  $1 \leq p < +\infty$ . Then there exists a positive constant  $C$ , depending only on  $p$  and  $\Omega$ , such that*

$$\|u\|_{L^p(\Omega, \gamma)} \leq C \|\nabla u\|_{L^p(\Omega, \gamma)} \quad (6.1)$$

for any  $u \in X = \left\{ u \in W^{1,p}(\Omega, \gamma) : \int_{\partial\Omega} u \varphi \, d\mathcal{H}^{N-1} = 0 \right\}$ .

Using (5.1) and (6.1) we obtain

**Corollary 6.1** *Let  $\Omega$  be a connected domain satisfying condition 2.1 and  $1 \leq p < +\infty$ . Then there exists a positive constant  $C$ , depending only on  $p$  and  $\Omega$ , such that*

$$\|Tu\|_{L^p(\partial\Omega, \gamma)} \leq C \|\nabla u\|_{L^p(\Omega, \gamma)} \quad (6.2)$$

for any  $u \in X$ .

**Remark 6.1** (*Application to PDE*) Let consider the eigenvalue problem

$$\begin{cases} -(u_{x_i} \varphi)_{x_i} = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial\Omega, \end{cases} \quad (6.3)$$

where  $\Omega$  is a connected domain satisfying condition 2.1. Arguing in a classical way using inequality (6.2) and the compactness of the trace operator, it is easy to prove that there exists an increasing sequence of eigenvalues of the problem (6.3) which tends to infinity. Moreover for  $\lambda_1 = 0$  the corresponding eigenvalue function  $u_1 = \text{const} \neq 0$  and the first nontrivial eigenvalue  $\lambda_2$  has the following characterization

$$\lambda_2 = \min \left\{ \frac{\|\nabla u\|_{L^2(\Omega, \gamma)}}{\|Tu\|_{L^2(\partial\Omega, \gamma)}}, u \in W^{1,2}(\Omega, \gamma) : \int_{\partial\Omega} u \varphi \, d\mathcal{H}^{N-1} = 0 \right\}.$$

## References

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